5.6 Matrix Exponentials and Linear Systems

Fundamental Matrix Solutions

The solution vectors of an $n \times n$ homogeneous linear system

$$\mathbf{x}' = \mathbf{A}\mathbf{x} \tag{1}$$

can be used to construct a square matrix $\mathbf{X} = \mathbf{\Phi}(t)$ that satisfies the matrix differential equation

$$\mathbf{X}' = \mathbf{A}\mathbf{X}.$$

Then the $n \times n$ matrix

$$\mathbf{\Phi}(t) = \begin{bmatrix} | & | & & | \\ \mathbf{x}_1(t) & \mathbf{x}_2(t) & \cdots & \mathbf{x}_n(t) \\ | & | & & | \end{bmatrix},$$

having these solution vectors as its column vectors, is called a **fundamental matrix** for the system in (1).

Example 1 Compute the fundamental matrix for the system

$$\mathbf{x}' = \begin{bmatrix} 4 & 2 \\ 3 & -1 \end{bmatrix} \mathbf{x}$$

We have $\mathbf{A} = \begin{bmatrix} 4 & 2 \\ 3 & -1 \end{bmatrix}$ with

eigenvalues $\lambda_1 = -2$ and $\lambda_2 = 5$ and eigenvectors $\mathbf{v}_1 = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

Thus we have two linearly independent solutions

$$\mathbf{x}_1(t) = \begin{bmatrix} 1 \\ -3 \end{bmatrix} e^{-2t} = \begin{bmatrix} e^{-2t} \\ -3e^{-2t} \end{bmatrix} \qquad \text{and} \qquad \mathbf{x}_2(t) = \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{5t} = \begin{bmatrix} 2e^{5t} \\ e^{5t} \end{bmatrix}$$

The fundamental matrix for the system is

$$\Phi(t) = \begin{bmatrix} \vec{x}_1(t) & \vec{x}_2(t) \end{bmatrix} = \begin{bmatrix} e^{-2t} & 2e^{5t} \\ -3e^{-2t} & e^{5t} \end{bmatrix}$$

which satisfies the equation of the matrices:

$$\Phi'(t) = \begin{bmatrix} 4 & 2 \\ 3 & -1 \end{bmatrix} \Phi(t).$$

Theorem 1. Fundamental Matrix Solutions

Let $\Phi(t)$ be a fundamental matrix for the homogeneous linear system $\mathbf{x}' = \mathbf{A}\mathbf{x}$. Then the [unique] solution of the initial value problem

$$\mathbf{x}' = \mathbf{A}\mathbf{x}, \qquad \mathbf{x}(0) = \mathbf{x}_0 \tag{2}$$

is given by

$$\mathbf{x}(t) = \mathbf{\Phi}(t)\mathbf{\Phi}(0)^{-1}\mathbf{x}_0. \tag{3}$$

In order to apply Eq. (3), we must be able to compute the inverse matrix $\mathbf{\Phi}(0)^{-1}$. The inverse of the nonsingular 2×2 matrix

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is

$$\mathbf{A}^{-1} = \frac{1}{\Delta}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix},$$

where $\Delta = \det(\mathbf{A}) = ad - bc \neq 0$.

Example 2 In **Example 1**, we have

$$\mathbf{x}' = \begin{bmatrix} 4 & 2 \\ 3 & -1 \end{bmatrix}\mathbf{x}$$

and $\mathbf{\Phi}(t) = \begin{bmatrix} e^{-2t} & 2e^{5t} \\ -3e^{-2t} & e^{5t} \end{bmatrix}$.

Find a solution satisfying the initial condition $\mathbf{x}_0 = \mathbf{x}(0) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

ANS: By Thm 1, we have

$$\vec{x}(t) = \underline{\Phi}(t)\underline{\Phi}^{-1}(0)\,\vec{x}_0$$

$$\underline{\Phi}(0) = \begin{bmatrix} e^{-2\cdot 0} & 2e^{5\cdot 0} \\ -3e^{-2\cdot 0} & e^{5\cdot 0} \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$a = 1$

$b = 2$

$c = -3$

$d = 1$

$$\underline{\Phi}^{-1}(0) = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{1+6}\begin{bmatrix} 1 & -2 \\ 3 & 1 \end{bmatrix} = \frac{1}{7}\begin{bmatrix} 1 & -2 \\ 3 & 1 \end{bmatrix}$$

$$\vec{x}(t) = \Phi(t)\,\Phi^{-1}(0)\,\vec{x}_0$$

$$= \frac{1}{7}\begin{bmatrix} e^{-2t} & 2e^{5t} \\ -3e^{-2t} & e^{5t} \end{bmatrix} \left(\begin{bmatrix} 1 & -2 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right)$$

$$= \frac{1}{7}\begin{bmatrix} e^{-2t} & 2e^{5t} \\ -3e^{-2t} & e^{5t} \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

$$= \frac{1}{7}\begin{bmatrix} 3e^{-2t} + 4e^{5t} \\ -9e^{-2t} + 2e^{5t} \end{bmatrix}$$

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \frac{3}{7}e^{-2t} + \frac{4}{7}e^{5t} \\ -\frac{9}{7}e^{-2t} + \frac{2}{7}e^{5t} \end{bmatrix}$$

Exponential Matrices

How to construct a fundamental matrix for the system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ directly from $\mathbf{A}$? (without solving for eigenvalues)

Recall that the solution of $x' = ax$ is $x(t) = e^{at}$.

Since $(e^{at})' = a e^{at}$

We now define exponentials of matrices in such a way that

$$\mathbf{X}(t) = e^{\mathbf{A}t}$$

is a matrix solution of the matrix differential equation

$$\mathbf{X}' = \mathbf{A}\mathbf{X}$$

with $n \times n$ coefficient matrix $\mathbf{A}$, which is an analog to the $x(t) = e^{at}$ is a solution of the equation $x' = ax$.

How do we define $e^{\mathbf{A}}$?

In calculus, we have

Eg: $e^2 = 1 + 2 + \frac{2^2}{2!} + \frac{2^3}{3!} + \cdots$

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots + \frac{z^n}{n!} + \cdots$$

Similarly, we have the following definition.

Definition Exponential matrix

If A is an $n \times n$ matrix, then the **exponential matrix** $e^{\mathbf{A}}$ is the $n \times n$ matrix defined by the series

$$e^{\mathbf{A}} = \mathbf{I} + \mathbf{A} + \frac{\mathbf{A}^2}{2!} + \cdots + \frac{\mathbf{A}^n}{n!} + \cdots, \tag{4}$$

where $\mathbf{I}$ is the identity matrix.

If $\boxed{\mathbf{AB} = \mathbf{BA}\text{, then } e^{\mathbf{A}+\mathbf{B}} = e^{\mathbf{A}}e^{\mathbf{B}}}$

Matrix Exponential Solutions

Theorem 2 Matrix Exponential Solutions

If $\mathbf{A}$ is an $n \times n$ matrix, then the solution of the initial value problem

$$\mathbf{x}' = \mathbf{A}\mathbf{x}, \qquad \mathbf{x}(0) = \mathbf{x}_0$$

is given by

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}_0,$$

and this solution is unique. Recall Thm 1. $\vec{x}(t) = \Phi(t)\Phi^{-1}(0)\vec{x}_0$

Idea of the proof:

- $(e^{At})' = Ae^{At}$ e^{At} satisfies $x' = Ax$
- $e^{At} = \Phi(t)$
- $e^{A \cdot 0} = I$

If we already know a fundamental matrix $\mathbf{\Phi}(t)$ for the linear system $\mathbf{x}' = \mathbf{A}\mathbf{x}$, then

$$\boxed{e^{\mathbf{A}t} = \mathbf{\Phi}(t)\mathbf{\Phi}(0)^{-1}.}$$

Example 3 Compute the matrix exponential $e^{\mathbf{A}t}$ for the system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ given in the problem.

$$\begin{aligned} x_1' &= 5x_1 - 4x_2 \\ x_2' &= 2x_1 - x_2 \end{aligned} \iff \begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 5 & -4 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

ANS: We will use $e^{At} = \Phi(t)\Phi^{-1}(0)$ to find e^{At}.

We first compute $\Phi(t)$

- We solve

$$0 = |A - \lambda I| = \begin{vmatrix} 5-\lambda & -4 \\ 2 & -1-\lambda \end{vmatrix} = (5-\lambda)(-1-\lambda)+8 = \lambda^2 - 4\lambda + 3 = (\lambda-1)(\lambda-3)$$

$\Rightarrow \lambda_1 = 1$ and $\lambda_2 = 3$.

- When $\lambda_1 = 1$, we solve $(A - \lambda_1 I)\vec{v}_1 = \vec{0}$.

$$\begin{bmatrix} 4 & -4 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow a - b = 0$$

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

- When $\lambda_2 = 3$, then we have

$$\begin{bmatrix} 2 & -4 \\ 2 & -4 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \Rightarrow a - 2b = 0$$

$$\vec{v}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Thus $\vec{x}_1(t) = \vec{v}_1 e^{\lambda_1 t} = \begin{bmatrix} e^t \\ e^t \end{bmatrix}$

$$\vec{x}_2(t) = \vec{v}_2 e^{\lambda_2 t} = \begin{bmatrix} 2e^{3t} \\ e^{3t} \end{bmatrix}$$

So $\Phi(t) = \begin{bmatrix} \vec{x}_1(t) & \vec{x}_2(t) \end{bmatrix} = \begin{bmatrix} e^t & 2e^{3t} \\ e^t & e^{3t} \end{bmatrix}$

$$\Phi(0) = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}, \quad \Phi^{-1}(0) = \frac{1}{1-2} \begin{bmatrix} 1 & -2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}$$

$$e^{At} = \Phi(t)\Phi^{-1}(0) = \begin{bmatrix} e^t & 2e^{3t} \\ e^t & e^{3t} \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} -e^t + 2e^{3t} & 2e^t - 2e^{3t} \\ -e^t + e^{3t} & 2e^t - e^{3t} \end{bmatrix}$$

Remark If $\mathbf{A}^n = \mathbf{0}$ for some positive integer n, then the exponential series in (4) terminates after a finite number of terms. Such a matrix—with a vanishing power—is said to be **nilpotent**.

Example 4 Show that the matrix $\mathbf{A}$ is nilpotent and then use this fact to find the matrix exponential $e^{\mathbf{A}t}$.

$$\mathbf{A} = \begin{bmatrix} 0 & 3 & 4 \\ 0 & 0 & 6 \\ 0 & 0 & 0 \end{bmatrix}$$

ANS: $A^2 = A \times A = \begin{bmatrix} 0 & 3 & 4 \\ 0 & 0 & 6 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 3 & 4 \\ 0 & 0 & 6 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 18 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$$A^3 = A^2 \cdot A = \begin{bmatrix} 0 & 0 & 18 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 3 & 4 \\ 0 & 0 & 6 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \mathbf{0}$$

So A is nilpotent.

Note

$$= \frac{A^3t^3}{3!} = 0$$

$$e^{At} = I + At + \frac{(At)^2}{2!} + \frac{(At)^3}{3!} + \frac{(At)^4}{4!} = 0 + \cdots \quad 0 \cdots 0$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 3 & 4 \\ 0 & 0 & 6 \\ 0 & 0 & 0 \end{bmatrix} t + \frac{1}{2} \begin{bmatrix} 0 & 0 & 18 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} t^2$$

$$e^{At} = \begin{bmatrix} 1 & 3t & 4t + 9t^2 \\ 0 & 1 & 6t \\ 0 & 0 & 1 \end{bmatrix}$$

Example 5 The coefficient matrix $\mathbf{A}$ in the following problem is the sum of a nilpotent matrix and a multiple of the identity matrix. Use this fact to solve the given initial value problem.

$$\mathbf{x}' = \underbrace{\begin{bmatrix} 2 & 5 \\ 0 & 2 \end{bmatrix}}_{A}\mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} 4 \\ 7 \end{bmatrix} = \vec{x}_0$$

$$B^2 = \begin{bmatrix} 0 & 5 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 5 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$$

ANS: By Thm 2, $\vec{x}(t) = \underline{e^{At}}\,\vec{x}_0$

$$A = \begin{bmatrix} 2 & 5 \\ 0 & 2 \end{bmatrix} = \underbrace{2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{\text{A multiple of the identity matrix}} + \underbrace{\begin{bmatrix} 0 & 5 \\ 0 & 0 \end{bmatrix}}_{\text{A nilpotent matrix}} = B$$

$$e^{At} = e^{(2I+B)t} = e^{2It+Bt} = e^{2It}e^{Bt} \quad (\text{since } IB = BI)$$

f $\mathbf{AB} = \mathbf{BA}$, then $e^{\mathbf{A}+\mathbf{B}} = e^{\mathbf{A}}e^{\mathbf{B}}$

$$e^{It} = I + It + \frac{\cancel{I^2}\,t^2}{2!} + \frac{\cancel{I^3}\,t^3}{3!} + \cdots = I\underbrace{\left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots\right)}_{e^t}$$

$$= e^t \cdot I$$

$$\boxed{e^{It} = e^t I}$$

So $e^{2It} = e^{2t} \cdot I$

$$e^{Bt} = \underline{I + Bt} + \frac{B^2 t^2}{2!} + \cdots$$

(with $B^2 = 0$)

$$= I + Bt = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 5t \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 5t \\ 0 & 1 \end{bmatrix}$$

Thus $e^{At} = e^{2It} \cdot e^{Bt}$

$$= \left(e^{2t} \cdot \cancel{I}\right)(I + Bt)$$

$$= e^{2t}(I + Bt)$$

$$e^{At} = e^{2t}\begin{bmatrix} 1 & 5t \\ 0 & 1 \end{bmatrix}$$

$$\vec{x}(t) = e^{At}\,\vec{x_0}$$

$$= e^{2t}\begin{bmatrix} 1 & 5t \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 4 \\ 7 \end{bmatrix}$$

$$= e^{2t}\begin{bmatrix} 4 + 35t \\ 7 \end{bmatrix}$$

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} e^{2t}(4 + 35t) \\ 7e^{2t} \end{bmatrix}$$